Confidence Intervals and Bias Corrections for the Stable32 Variance Functions

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Introduction

This paper describes the methods used by the Stable32 program for setting confidence intervals, showing error bars and making bias corrections in its variance functions. In general, this process involves determining the dominant power law noise type, calculating the equivalent number of Chi-squared degrees of freedom, establishing single or double-sided confidence limits and setting the error bars accordingly. In addition, some of the Stable32 statistics have biases that are corrected to agree with the expected value of their corresponding Allan variance. A summary of these methods is shown in the table at the end of this paper.

Power Law Noise Identification

It is necessary to identify the spectral characteristics of the dominant noise process in order to set the confidence interval, show the error bars and, where appropriate, apply a bias correction to the results of a time domain frequency stability analysis.

- **Noise Spectra**

The random phase and frequency fluctuations of a frequency source can be modeled by power law spectral densities of the form:

$$S_y(f) = h_\alpha f^{\alpha}$$

where: $S_y(f)$ = one-sided power spectral density of the fractional frequency fluctuations, 1/Hz
F = Fourier or sideband frequency, Hz
h_α = intensity coefficient
α = exponent of the power law noise process.

The exponent of the power law noise process, α, is closely related to μ, the slope of the log-log dependence of the Allan variance, $\sigma^2_y(\tau) = A_\mu \tau^\mu$.

- **Power Law Noise Processes**

The spectral characteristics of the power law noise processes commonly used to describe the performance of frequency sources are shown in the following table:

Noise Type	α	μ
White PM	2	-2
Flicker PM	1	-2
White FM	0	-1
Flicker FM	-1	0
Random Walk FM	-2	1
Flicker Walk FM	-3	2
Random Run FM	-4	3

- **Power Law Noise Identification**

The dominant power law noise type can be estimated by comparing the ratio of the N-sample (standard) variance to the 2-sample (Allan) variance of the data (the B1 bias factor, see below) to the value expected of this ratio for the pure noise types (for the same averaging factor). This method of noise identification, while not perfect, is reasonably effective in most cases. The main limitations are (1) its inability to distinguish between white and flicker PM noise, and (2) its limited precision at large averaging factors where there are few analysis points. The former limitation can be overcome by supplementing the B1 ratio test with one based on R(n), the ratio of the modified Allan variance to the normal Allan variance (see below). That technique is applied to members of the modified family of variances (MVAR, TVAR, and MTOT). The second limitation is avoided by using the previous noise type estimate at the longest averaging time of an analysis run. One further limitation is that the R(n) ratio is not meaningful at a unity averaging factor.

Confidence Intervals

- **Confidence Intervals**

The confidence interval of a variance estimate depends on the variance type (normal Allan, overlapping Allan, modified Allan, time, total, modified total, Hadamard and overlapping Hadamard), the nominal value, the number of data points and averaging factor, the statistical confidence factor desired, and the type of noise. Stable32 uses χ^2 for setting the confidence intervals and error bars in its stability analysis and plotting functions.

- **Chi-Squared Confidence Intervals**

Sample variances are distributed according to the expression:

$$\chi^2 = \frac{\text{edf} \cdot s^2}{\sigma^2},$$

where χ^2 is the Chi-square, s^2 is the sample variance, σ^2 is the true variance, and edf is the equivalent number of degrees of freedom (not necessarily an integer). The edf is determined by the number of analysis points and the noise type. The Stable32 program includes procedures for establishing single or double-sided confidence intervals with a selectable confidence factor, based on χ^2 statistics, for many of its variance functions. The general procedure is to choose a single or double-limited confidence factor, p, calculate the corresponding χ^2 value, determine the edf from the variance type, noise type and number of analysis points, and thereby set the statistical limit(s) on the variance. For a double-sided limits

$$\sigma^2_{min} = s^2 \cdot \frac{\text{edf}}{\chi^2(p, \text{edf})} \quad \text{and} \quad \sigma^2_{max} = s^2 \cdot \frac{\text{edf}}{\chi^2(1-p, \text{edf})}.$$

- **Overlapping Samples**

Improved statistical confidence can be obtained by making better use of the available data by forming all possible frequency 1st differences (or phase 2nd differences) for a given averaging time. Although these fully overlapping samples are not all statistically independent, they nevertheless help to improve the confidence of the resulting Allan deviation estimate.

- **Extended Samples**

Even better confidence can be obtained by extending the data set to form even more samples. That technique is used by the total and modified total variance. A different method of forming all possible samples is used for the Thêo1 statistic.

- **AVAR, MVAR, TVAR and HVAR EDF**

The equivalent number of χ^2 degrees of freedom (edf) for the Allan variance (AVAR), the modified Allan variance (MVAR) and the related time variance (TVAR), and the Hadamard variance (HVAR) is found by a combined algorithm developed by C.A. Greenhall based on its generalized autocovariance function [13].

- **TOTVAR and TTOT EDF**

The edf for the total variance (TOTVAR) and the related total time variance (TTOT) is given by the formula $b(T/\tau)$ -c, where T is the length of the data record, τ is the averaging time, and b & c are coefficients that depend on the noise type as shown in the following table:

Power Law Noise Type	TOTVAR edf Coefficients	
	b	c
White FM	1.50	0
Flicker FM	1.17	0.22
Random Walk FM	0.93	0.36

- **MTOT EDF**

The edf for the modified total variance (MTOT) is given by the same formula $b(T/\tau)$ -c, where T is the length of the data record, τ is the averaging time, and b & c are coefficients that depend on the noise type as shown in the following table:

Power Law Noise Type	MTOT edf Coefficients	
	b	c
White PM	1.90	2.10
Flicker PM	1.20	1.40
White FM	1.10	1.20
Flicker FM	0.85	0.50
Random Walk FM	0.75	0.31

- **HTOT EDF**

The edf for the Hadamard total variance (HTOT) is given by the method described by D. Howe et al [11]. It applies to $-4 \leq \alpha \leq 0$, and uses the formula $(T/\tau)/(b_0+b_1\tau/T)$, where T is the length of the data record, τ is the averaging time, and b_0 & b_1 are coefficients that depend on the noise type as shown in the following table:

Power Law Noise Type	HTOT edf Coefficients	
	b_0	b_1
White FM	0.559	1.004
Flicker FM	0.868	1.140
Random Walk FM	0.938	1.696
Flicker Walk FM	2.554	0.974
Random Run FM	3.149	1.276

- **Thêo1 EDF**

The equivalent number of χ^2 degrees of freedom (edf) for the Thêo1 variance is determined by the following approximation formulae for each power low noise type.

Power Law NoiseType	Thêo1 edf, where N = # phase data points, τ=0.75m, m = averaging factor = τ/τ_0
White PM	$edf = \left(\frac{0.86(N_x+1)(N_x-\frac{4}{3}\cdot t)}{N_x-t}\right)\left(\frac{t}{t+1.14}\right)$
Flicker PM	$edf = \left(\frac{4.798N_x^2-6.374N_x t+12.387t}{(t+36.6)^{1/2}(N_x-t)}\right)\left(\frac{t}{t+0.3}\right)$
White FM	$edf = \left[\frac{4.1N_x+0.8}{t}-\frac{3.1N_x+6.5}{N_x}\right]\left(\frac{t^{3/2}}{t^{3/2}+5.2}\right)$
Flicker FM	$edf = \left(\frac{2N_x^2-1.3N_x t-3.5t}{N_x t}\right)\left(\frac{t^3}{t^3+2.3}\right)$
Random Walk FM	$edf = \left(\frac{4.4N_x-2}{2.9t}\right)\left(\frac{(4.4N_x-1)^2-8.6t(4.4N_x-1)+11.4t^2}{(4.4N_x-3)^2}\right)$

- **# Analysis Points**

The # in the Run stability table, and the # Analysis Pts in the detailed Sigma dialog box, is the number of analysis points (the # of 2nd or 3rd differences summed) in the sigma calculation. This number is used in determining the confidence intervals. Without gaps, the #s are equal to the following:

Sigma Type	#
Normal Allan	M/m -1
Overlapping Allan	M-2m+1 = N-2m
Modified & Time	M-3m+2 = N-3m+1
Total	M-m = N-m-1
Mod Total & Time Total	M-3m+2 = N-3m+1
Hadamard	M/m -2
Overlapping Hadamard	M-3m+1 = N-3m
Hadamard Total	M-3m+1=N-3m
Thêo1	M·m/2 = (N-1)(m/2)

where: N = # phase data points = M+1
M = # frequency data points
m = AF = averaging factor

For the normal Allan and Hadamard variances, the rounding down associated with successive averaging may reduce the #. With gaps, the # can depend on exactly where the gaps are. In all cases, the # is counted dynamically during the calculation as the statistical sums are accumulated.

Bias Functions

- **Bias Functions**

Several bias functions are defined and used in the analysis of frequency stability. These bias functions are defined below. The Stable32 program uses the B1, the standard variance to Allan variance ratio, and R(n), the modified Allan variance to normal Allan variance ratio bias functions for the identification of noise types.

- **B1 Bias Function**

The B1 bias function is the ratio of the N-sample (standard) variance to the 2-sample (Allan) variance with dead time ratio $r = T/\tau$, where T = time between measurements, τ=averaging time, and μ=exponent of τ in Allan variance for a certain power law noise process:

$$B1(N, r, \mu) = \frac{\sigma^2(N, T, \tau)}{\sigma^2(2, T, \tau)}.$$

- **B2 Bias Function**

The B2 bias function is the ratio of the 2-sample (Allan) variance with dead time ratio $r = T/\tau$ to the 2-sample (Allan) variance without dead time (r = 1):

$$B2(r, \mu) = \frac{\sigma^2(2, T, \tau)}{\sigma^2(2, \tau, \tau)}.$$

- **B3 Bias Function**

The B3 bias function is the ratio of the N-sample (standard) variance with dead time ratio $r = T/\tau$ at multiples $M = \tau/\tau_0$ of the basic averaging time τ_0 to the N-sample variance with the same dead time ratio at averaging time τ:

$$B3(N, M, r, \mu) = \frac{\sigma^2(N, M, T, \tau)}{\sigma^2(N, T, \tau)}.$$

- **R(n) Function**

The R(n) function is the ratio of the modified Allan variance to the normal Allan variance for n = # phase data points. Note: R(n) is also a function of α, the exponent of the power law noise type:

$$R(n) = \frac{\text{Mod}\,\sigma^2_y(\tau)}{\sigma^2_y(\tau)}.$$

- **TOTVAR Bias Function**

The TOTVAR statistic is an unbiased estimator of the Allan variance for white and flicker PM noise, and for white FM noise. For flicker and random walk FM noise, TOTVAR is biased low as τ becomes significant compared with the record length. The ratio of the expected value of TOTVAR to AVAR is given by the expression:

$$B(\text{TOTAL}) = 1 - a\cdot(\tau/T), \quad 0 < \tau \leq T/2$$

where $a = 1/3\cdot\ln 2 = 0.481$ for flicker FM noise, $a = 3/4 = 0.750$ for random walk FM noise, and T is the record length. At the maximum allowable value of $\tau = T/2$, TOTVAR is biased low by about 24% for RW FM noise. The Stable32 program applies this bias function automatically to correct the reported TOTVAR result.

- **MTOT and TTOT Bias Function**

The MTOT statistic is a biased estimator of the modified Allan variance. The MTOT bias factor (the ratio of the expected value of Mod Totvar to MVAR) depends on the noise type but is essentially independent of the averaging factor and # of data points, as shown in the following table:

Noise Type	Bias Factor
W PM	0.94
F PM	0.83
W FM	0.73
F FM	0.70
RW FM	0.69

Stable32 program applies this bias function automatically to correct the reported MTOT result.

- **HTOT Bias Function**

The HTOT statistic is a biased estimator of the Hadamard variance. The HTOT bias factor (the ratio of the expected value of the Total HVAR to HVAR) depends on the noise type but is essentially independent of the averaging factor and # of data points, as shown in the following table [11]:

Noise Type	Bias Factor
W FM	0.995
F FM	0.851
RW FM	0.771
FW FM	0.717
RR FM	0.679

Stable32 program applies this bias function automatically to correct the reported MTOT result.

- **Thêo1 Bias Function**

The Thêo1 statistic is an unbiased estimator of the Allan variance for white FM noise. For other power law noise types, the following bias corrections should be applied to the Thêo1 deviation.

Noise Type	Bias Factor
W PM	0.63
F PM	0.77
W FM	1.00
F FM	1.31
RW FM	1.50

Stable32 program applies this bias function according to the selected noise. The program can also perform an Auto Match in its combined ADEV and Thêo1 plot to automatically match the Thêo1 result to ADEV regardless of the noise type.

- **Summary of Stable32 EDF and Bias Functions**

The following table summarizes the use of the equivalent degrees of freedom, confidence intervals and bias corrections in the variance calculations of the Stable32 Sigma and Run functions:

<table>
<tr><th colspan="5">Stable32 Noise ID, EDF, CI & Bias Methods</th></tr>
<tr><th>Variance Type</th><th>Noise ID Method</th><th>EDF Calc</th><th>CI/Error Bars</th><th>Bias Calc</th></tr>
<tr><td>Normal Allan (AVAR)</td><td rowspan="2">B1 ratio</td><td rowspan="6">Combined EDF [13]</td><td rowspan="11">χ^2</td><td rowspan="6">N/A</td></tr>
<tr><td>Overlapping Allan (AVAR)</td></tr>
<tr><td>Modified Allan (MVAR)</td><td rowspan="2">B1 plus R(n) for α = 1,2</td></tr>
<tr><td>Time (TVAR)</td></tr>
<tr><td>Hadamard (HVAR)</td><td>N/A</td></tr>
<tr><td>Overlapping Hadamard (HVAR)</td><td>B1 ratio</td></tr>
<tr><td>Hadamard Total (HTOT)</td><td></td><td>[11]</td><td></td></tr>
<tr><td>Total (TOTVAR)</td><td>B1 ratio</td><td>TOTEDF [4]</td><td>TOTVAR Bias [6]</td></tr>
<tr><td>Mod Total (MTOT)</td><td rowspan="2">B1 plus R(n) for α = 1,2</td><td rowspan="2">MTOTEDF for m>8, else EDF [5], [12]</td><td rowspan="2">MTOT Bias [7]</td></tr>
<tr><td>Time Total (TTOT)</td></tr>
<tr><td>Thêo1</td><td>Match to ADEV</td><td>[12]</td><td>[12]</td></tr>
<tr><td>TIE rms</td><td>N/A</td><td>N/A</td><td>None</td><td>N/A</td></tr>
<tr><td>MTIE</td><td>N/A</td><td>N/A</td><td>None</td><td>N/A</td></tr>
</table>

• References

The following references apply to the edf and bias functions used in Stable32:

1. "IEEE Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology - Random Instabilities", *IEEE Std 1139-1999*, July 1999.

2. C.A. Greenhall, "Estimating the Modified Allan Variance", *Proc. IEEE 1995 Freq. Contrl. Symp.*, pp. 346-353, May 1995.

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4. C.A. Greenhall, private communication, May 1999.

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10. J.A. Barnes and D.W. Allan, Variances Based on Data with Dead Time Between the Measurements", *NIST Technical Note 1318,* 1990.

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12. D.A. Howe and T.K. Peppler, “Estimation of Very Long-Term Frequency Stability Using a Special-Purpose Statistic”, ", *Proc. 2003 Joint Meeting of the European Freq. and Time Forum and the IEEE International Freq. Contrl. Symp*., May 2003, (to be published).

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